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A Weak Formulation of the Boltzmann Equation Based on the Fourier Transform

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Abstract In this article we present an alternative formulation of the spatially homogeneous Boltzmann equation. Rewriting the weak form of the equation with shifted test functions and using Fourier techniques, it turns out that the transformed problem contains only a three-fold integral.

Explicit formulas for the transformed collision kernel are presented in the case of VHS models for hard and soft potentials. For isotropic Maxwellian molecules, a classical result by Bobylev is recovered, too.

Keywords Kinetic theory of gases · Boltzmann equation

1 Introduction

The initial value problem for the spatially homogeneous Boltzmann equation reads

$$\begin{aligned} f_t(t,v) &= Q(f,f)(t,v) \quad \text{for } t>0,\ v\in\mathbb{R}^3, \\ f(0,v) &= f_0(v) \quad \text{for } v\in\mathbb{R}^3, \end{aligned} \tag{1}$$

where the unknown distribution density function f depends on time $t \geq 0$ and velocity $v \in \mathbb{R}^3$. The collision operator (or collision integral) Q in bilinear form is given by

$$Q(f,g)(v) = \int_{\mathbb{R}^3}\int_{S^2} B(v,w,e)(f(v')g(w') - f(v)g(w))de\,dw, \tag{2}$$

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where S^2 denotes the unit sphere in $\mathbb{R}^3$. The velocity transformation $(v, w) \mapsto (v', w')$ is defined as follows:

$$v' = \frac{1}{2}(v + w + |v - w|e), \qquad w' = \frac{1}{2}(v + w - |v - w|e). \tag{3}$$

The so-called collision kernel B describes the microscopic details of the particle interaction and is generally assumed to be of the form

$$B(v, w, e) = b_\lambda(\mu)|v - w|^\lambda, \qquad \mu = \frac{(v - w, e)}{|v - w|}, \quad -3 < \lambda \leq 1, \tag{4}$$

where $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^3$. In general, the angular part b_λ contains a non-integrable singularity at $\mu = 1$. In practical applications, the so-called VHS model for hard potentials is frequently considered, i.e. the function b_λ is assumed to be constant:

$$B(v, w, e) = C_\lambda |v - w|^\lambda, \quad 0 \leq \lambda \leq 1. \tag{5}$$

We shall benefit from the following weak formulation of the collision operator: For any appropriate test function φ, it holds

$$\begin{aligned} &\int_{\mathbb{R}^3} Q(f, g)(v)\varphi(v)dv \\ &\quad = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v - w|^\lambda f(v)g(w) \\ &\qquad \times \int_{S^2} b_\lambda(\mu)(\varphi(v') + \varphi(w') - \varphi(v) - \varphi(w))de\,dw\,dv. \end{aligned} \tag{6}$$

Theoretically and numerically, the Boltzmann equation constitutes a great challenge and there is a need to find alternative formulations of the problem which are more accessible to investigation and numerical solution. An important step towards that direction was made by Bobylev in [5], where he found that in the case of Maxwellian molecules, the Fourier transform of the equation acquires the form

$$\hat{f}_t(\xi) = \int_{S^2} b_0(\mu)\left(\hat{f}\left(\frac{\xi + |\xi|e}{2}\right)\hat{f}\left(\frac{\xi - |\xi|e}{2}\right) - \hat{f}(\xi)\hat{f}(0)\right)de, \tag{7}$$

noting that here $\mu = (\xi, e)/|\xi|$. This remarkable result could be used to find exact solutions to the initial value problem (1).

Since then, Fourier techniques have proven to be useful to obtain theoretical and numerical results. We briefly quote only a few of the corresponding literature and refer the reader to [9] for a survey on the matter including an extensive list of references.

Regularity properties were obtained in [8, 11, 15, 16, 18, 26] for the so-called cut-off collision kernels (i.e. b_λ is assumed to be integrable over S^2). Smoothness issues in the general case (4) were investigated in [1, 25]. We also mention [10], where it was proven for a class of non cut-off kernels that the solution to (1) is a Schwartz function for all $t > 0$ and [3], where the formula (7) was the basis to show that the (weak) solution to (1) is C^∞.

As for the numerics, several ways to use the FFT algorithm for efficient and accurate computation of an approximate solution were proposed: A direct use of (7) was presented in [6]. In [19], a Fourier Galerkin spectral method was introduced which approximates f by a truncated Fourier series. Further developments of this approach were presented in [17, 20, 21]. A detailed study for the case of non-cut-off kernels can be found in [22]. Alternative approaches rely on the rewriting of the collision operator such that the FFT algorithm leads to a significant reduction of numerical cost. Examples for this can be found in [7, 13].

In the spirit of these works, our goal is to rewrite the initial value problem for the spatially homogeneous Boltzmann equation in such a way that the resulting problem has a significantly simpler structure than the original one. This can be achieved by appropriate use of the Fourier transform as was previously done in the work [2], where a semi-explicit representation of the transformed collision operator, known up to the Fourier transform of the collision kernel B, was derived. In order to obtain fully explicit expressions however, we will focus on VHS models and by nature, our formulas are closely related to the one in [2] and are similar in structure to those occurring when using the Fourier spectral approximation.

After completion of our work we became aware of the preprint [14] where expansion techniques were successfully used to represent the Fourier transform of a solution to the Boltzmann equation.

The rest of the paper is organized as follows. We conclude this introductory section with some notation used later. In Sect. 2, we derive a transformed version of the weak formulation of the Boltzmann equation which involves a three-dimensional replacement for the five-fold collision integral. In the following section, we compute the transformed collision kernel for the VHS model and it turns out that explicit expressions can be obtained not only for hard potentials as given by (5), but for all

$$B(v,w,e) = C_\lambda |v-w|^\lambda, \quad -3 < \lambda \le 1. \tag{8}$$

In the sequel, we recover Bobylev's classical result for Maxwellian molecules with constant angular part. Finally, we draw some conclusions.

We will denote by φ any test function in the Schwartz space $\mathbb{S}$ of infinitely smooth and rapidly decreasing functions on $\mathbb{R}^3$. The Fourier transform of φ is denoted by

$$\hat{\varphi}(\xi) = \mathcal{F}_{v\to\xi}(\varphi)(\xi) = \int_{\mathbb{R}^3} \varphi(v) \mathrm{e}^{\mathrm{i}(v,\xi)} dv,$$

and the inverse Fourier transform is

$$\varphi(v) = \mathcal{F}^{-1}_{\xi\to v}(\hat{\varphi})(v) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \hat{\varphi}(\xi) \mathrm{e}^{-\mathrm{i}(v,\xi)} d\xi.$$

We will also denote by

$$(\Phi, \varphi) = \int_{\mathbb{R}^3} \Phi(v)\varphi(v) dv$$

the action of a tempered distribution $\Phi \in \mathbb{S}'$ on a test function φ. We recall that the convolution of a tempered distribution Φ with a test function $\varphi \in \mathbb{S}$ is again a tempered distribution and that the following well-known identity holds (cf. [23]):

$$\widehat{(\Phi * \varphi)}(\xi) = \hat{\Phi}(\xi)\hat{\varphi}(\xi).$$

2 Transformation of the Boltzmann Equation

Let us consider the following weak formulation of the spatially homogeneous Boltzmann equation: For $z \in \mathbb{R}^3$, we demand that (1) is fulfilled in the sense of tempered distributions, i.e.

$$\forall \varphi \in \mathbb{S} : (f_t, \varphi(z-\cdot)) = (Q(f,f), \varphi(z-\cdot)). \tag{9}$$

To replace $\varphi(v)$ by the shifted version $\varphi(z-v)$ is crucial for our approach, as will be clear later. We start with the right-hand side of (9) and define more generally

$$(J(f,g),\varphi)(z) = (Q(f,g), \varphi(z-\cdot)). \tag{10}$$

Using the identity (6), we obtain

$$\begin{aligned}(J(f,g),\varphi)(z) &= \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v-w|^\lambda f(v)g(w) \\ &\quad \times \int_{S^2} b_\lambda(\mu)(\varphi(z-v') + \varphi(z-w') - \varphi(z-v) - \varphi(z-w)) de dw dv \\ &= \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v-w|^\lambda f(z-v)g(z-w) \\ &\quad \times \int_{S^2} b_\lambda(\mu)(\varphi(v') + \varphi(w') - \varphi(v) - \varphi(w)) de dw dv,\end{aligned} \tag{11}$$

where we replaced the integration variables v and w by $z-v$ and $z-w$, respectively, and e by $-e$.
With the notation

$$T_\lambda(\varphi)(v,w) = \frac{|v-w|^\lambda}{2}\int_{S^2} b_\lambda(\mu)(\varphi(v') + \varphi(w') - \varphi(v) - \varphi(w)) de, \tag{12}$$

we can write (11) as

$$(J(f,g),\varphi)(z) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(z-v)g(z-w)T_\lambda(\varphi)(v,w) dw dv. \tag{13}$$

Assuming that f and g are inverse images under the Fourier transform,

$$\begin{aligned} f(z-v) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \hat{f}(\xi_1) \mathrm{e}^{-\mathrm{i}(z-v,\xi_1)} d\xi_1, \\ g(z-w) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \hat{g}(\xi_2) \mathrm{e}^{-\mathrm{i}(z-w,\xi_2)} d\xi_2, \end{aligned}$$

we derive from (13)

$$(J(f,g),\varphi)(z) = \frac{1}{(2\pi)^6}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \hat{f}(\xi_1)\hat{g}(\xi_2)\mathrm{e}^{-\mathrm{i}(z,\xi_1+\xi_2)} \widehat{T_\lambda(\varphi)}(\xi_1,\xi_2) d\xi_2 d\xi_1, \tag{14}$$

where

$$\widehat{T_\lambda(\varphi)}(\xi_1,\xi_2) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} T_\lambda(\varphi)(v,w)\mathrm{e}^{\mathrm{i}(v,\xi_1)}\mathrm{e}^{\mathrm{i}(w,\xi_2)} dw dv \tag{15}$$

is the six-dimensional Fourier Transform of $T_\lambda(\varphi)$.

The expression (15) can be simplified by the following change of integration variables

$$x = \frac{v+w}{2}, \qquad y = \frac{v-w}{2} \quad \Rightarrow \quad dwdv = 8dxdy,$$

such that we have now

$$\mu = \frac{(y,e)}{|y|} \tag{16}$$

and

$$\begin{aligned}\widehat{T_\lambda(\varphi)}(\xi_1,\xi_2) = 2^{\lambda+2}\int_{S^2}\int_{\mathbb{R}^3} b_\lambda(\mu)|y|^\lambda \mathrm{e}^{\mathrm{i}(y,\xi_1-\xi_2)} \\ \times \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}(x,\xi_1+\xi_2)}(\varphi(x+|y|e)+\varphi(x-|y|e)-\varphi(x+y) \\ -\varphi(x-y))dxdyde.\end{aligned} \tag{17}$$

We treat the four summands in the innermost integral in (17) separately, using the change of integration variables

$$u = x+|y|e, \qquad u = x-|y|e, \qquad u = x+y, \qquad u = x-y,$$

respectively. In any case it holds $dx = du$ and we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}(x,\xi_1+\xi_2)}(\varphi(x+|y|e)+\varphi(x-|y|e)-\varphi(x+y)-\varphi(x-y))dx \\ &\quad = \hat{\varphi}(\xi_1+\xi_2)(\mathrm{e}^{-\mathrm{i}(|y|e,\xi_1+\xi_2)}+\mathrm{e}^{\mathrm{i}(|y|e,\xi_1+\xi_2)}-\mathrm{e}^{-\mathrm{i}(y,\xi_1+\xi_2)}-\mathrm{e}^{\mathrm{i}(y,\xi_1+\xi_2)}).\end{aligned}$$

Inserting this into (17) and using for (14) the change of variables

$$\xi = \xi_1+\xi_2, \qquad \eta = \xi_1-\xi_2 \quad \Rightarrow \quad d\xi_2 d\xi_1 = \frac{1}{8}d\eta d\xi,$$

we have

$$\begin{aligned}(J(f,g),\varphi)(z) &= \frac{2^{\lambda-1}}{(2\pi)^6}\int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}(z,\xi)}\hat{\varphi}(\xi)\int_{\mathbb{R}^3} \hat{f}\left(\frac{\xi+\eta}{2}\right)\hat{g}\left(\frac{\xi-\eta}{2}\right)\widehat{T_\lambda}(\xi,\eta)d\eta d\xi \\ &= \frac{2^{\lambda-1}}{(2\pi)^3}\mathcal{F}^{-1}_{\xi\to z}\left(\hat{\varphi}(\cdot)\int_{\mathbb{R}^3} \hat{f}\left(\frac{\cdot+\eta}{2}\right)\hat{g}\left(\frac{\cdot-\eta}{2}\right)\widehat{T_\lambda}(\cdot,\eta)d\eta\right)(z)\end{aligned}$$

with the kernel

$$\begin{aligned}\widehat{T_\lambda}(\xi,\eta) = \int_{S^2}\int_{\mathbb{R}^3} b_\lambda(\mu)|y|^\lambda \mathrm{e}^{\mathrm{i}(y,\eta)} \\ \times (\mathrm{e}^{-\mathrm{i}(|y|e,\xi)}+\mathrm{e}^{\mathrm{i}(|y|e,\xi)}-\mathrm{e}^{-\mathrm{i}(y,\xi)}-\mathrm{e}^{\mathrm{i}(y,\xi)})dyde,\end{aligned} \tag{18}$$

which is independent of the test function φ and contains the information about the particle interaction. Recall that $y=(v-w)/2$ and therefore, the infinitely smooth term in brackets is sufficient to cancel out the singularities in the collision kernel (cf. [4, 22, 24] for details).

The meaning of the parameter z is the following: Applying the Fourier transform $\mathcal{F}_{z\to\xi}$ to both sides of the weak formulation (9) and using (18), we see that the transformed Boltzmann equation reads

$$\hat{f}_t(t,\xi)=\frac{2^{\lambda-1}}{(2\pi)^3}\int_{\mathbb{R}^3}\hat{f}\left(t,\frac{\xi+\eta}{2}\right)\hat{f}\left(t,\frac{\xi-\eta}{2}\right)\widehat{T_\lambda}(\xi,\eta)d\eta, \tag{19}$$

where $\widehat{T_\lambda}$ is given by (18). Note that the five-fold integral is reduced to a three-fold one having a very simple structure. The equation is to be understood in the weak sense, i.e.

$$\forall\psi\in\mathbb{S}:(\hat{f}_t-\mathcal{J}_\lambda(\hat{f},\hat{f}),\hat{\psi})=0, \tag{20}$$

with the transformed collision operator acting on the Fourier transform of f:

$$\mathcal{J}_\lambda(\hat{f},\hat{f})(\xi)=\frac{2^{\lambda-1}}{(2\pi)^3}\int_{\mathbb{R}^3}\hat{f}\left(\frac{\xi+\eta}{2}\right)\hat{f}\left(\frac{\xi-\eta}{2}\right)\widehat{T_\lambda}(\xi,\eta)d\eta. \tag{21}$$

Remark 1 It is easy to check that

$$\widehat{T_\lambda}(0,\eta)=0,\qquad\nabla_\xi\widehat{T_\lambda}(0,\eta)=0,\qquad\Delta_\xi\widehat{T_\lambda}(0,\eta)=0.$$

Taking formally the corresponding derivatives on both sides of the transformed equation, it follows that

$$\hat{f}(0),\qquad \mathrm{i}\nabla\hat{f}(0),\qquad -\frac{1}{2}\Delta\hat{f}(0)$$

are conserved quantities. But this corresponds to the conservation of mass, momentum and energy in terms of the original equation (1).

3 The Kernel $\widehat{T_\lambda}$ for the VHS Model

Our goal in this section is to compute explicit expressions for the kernel T_λ in (18) under the assumption that the collision kernel is of the form (8). We have

$$\begin{aligned}\widehat{T_\lambda}(\xi,\eta)=C_\lambda\int_{\mathbb{R}^3}&|y|^\lambda \mathrm{e}^{\mathrm{i}(y,\eta)}\\&\times\int_{S^2}(\mathrm{e}^{-\mathrm{i}(|y|e,\xi)}+\mathrm{e}^{\mathrm{i}(|y|e,\xi)}-\mathrm{e}^{-\mathrm{i}(y,\xi)}-\mathrm{e}^{\mathrm{i}(y,\xi)})de\,dy.\end{aligned} \tag{22}$$

The following identity will be useful for this task: For $\zeta\in\mathbb{R}^3$ it holds

$$\int_{S^2}\mathrm{e}^{\mathrm{i}(e,\zeta)}de=4\pi\frac{\sin(|\zeta|)}{|\zeta|}, \tag{23}$$

as can be easily verified by choosing a suitable parametrization of S^2.

3.1 The Case $-3 < \lambda \le 1$, $\lambda \neq 0$

We start the computation of (22) with the integral over S^2 using (23):

$$\int_{S^2} (\mathrm{e}^{-\mathrm{i}(|y|e,\xi)} + \mathrm{e}^{\mathrm{i}(|y|e,\xi)})de = 8\pi \frac{\sin(|y||\xi|)}{|y||\xi|}.$$

Inserting this into (22), using the notation

$$r = |y|, \qquad e_y = \frac{y}{|y|},$$

and applying (23) repeatedly, we obtain

$$\begin{aligned}
\widehat{T_\lambda}(\xi,\eta) &= 4\pi C_\lambda \int_{\mathbb{R}^3} |y|^\lambda \mathrm{e}^{\mathrm{i}(y,\eta)} \left(\frac{2\sin(|y||\xi|)}{|y||\xi|} - \mathrm{e}^{-\mathrm{i}(y,\xi)} - \mathrm{e}^{\mathrm{i}(y,\xi)} \right) dy \\
&= 4\pi C_\lambda \left(\int_{\mathbb{R}^3} |y|^\lambda \mathrm{e}^{\mathrm{i}(y,\eta)} \frac{2\sin(|y||\xi|)}{|y||\xi|} dy - \int_{\mathbb{R}^3} |y|^\lambda (\mathrm{e}^{\mathrm{i}(y,\eta-\xi)} + \mathrm{e}^{\mathrm{i}(y,\eta+\xi)}) dy \right) \\
&= 4\pi C_\lambda \left(\frac{2}{|\xi|} \int_0^\infty r^{\lambda+1} \sin(r|\xi|) \int_{S^2} \mathrm{e}^{\mathrm{i}(re_y,\eta)} de_y dr \right. \\
&\qquad \left. - \int_0^\infty r^{\lambda+2} \int_{S^2} (\mathrm{e}^{\mathrm{i}(re_y,\eta-\xi)} + \mathrm{e}^{\mathrm{i}(re_y,\eta+\xi)}) de_y dr \right) \\
&= 16\pi^2 C_\lambda \left(\frac{2}{|\xi||\eta|} \int_0^\infty r^\lambda \sin(r|\xi|)\sin(r|\eta|) dr \right. \\
&\qquad \left. - \left(\frac{1}{|\xi-\eta|} \int_0^\infty r^{\lambda+1} \sin(r|\xi-\eta|) dr + \frac{1}{|\xi+\eta|} \int_0^\infty r^{\lambda+1} \sin(r|\xi+\eta|) dr \right) \right).
\end{aligned}$$

The Fourier transforms occurring in these expressions are well-known (cf. e.g. [12]). In fact, the following result holds for all $\lambda \in (-3, 1]$:

$$\begin{aligned}
\widehat{T_\lambda}(\xi,\eta) &= -16\pi^2 C_\lambda \Gamma(\lambda+1) \sin\left(\frac{\pi\lambda}{2}\right) \\
&\quad \times \left(\frac{||\xi| - |\eta||^{-\lambda-1} - ||\xi| + |\eta||^{-\lambda-1}}{|\xi||\eta|} \right. \\
&\quad \left. - (\lambda+1) \left(\frac{1}{|\xi-\eta|^{\lambda+3}} + \frac{1}{|\xi+\eta|^{\lambda+3}} \right) \right).
\end{aligned} \tag{24}$$

For the special case $\lambda = 1$, which corresponds to the hard spheres interaction model, we have

$$\widehat{T_1}(\xi,\eta) = -32\pi^2 C_1 \left(\frac{2}{||\xi|^2 - |\eta|^2|^2} - \left(\frac{1}{|\xi-\eta|^4} + \frac{1}{|\xi+\eta|^4} \right) \right).$$

Inserted into (19), the transformed equation for the hard sphere model reads:

$$\begin{aligned}
\hat{f}_t(\xi) &= -\frac{4C_1}{\pi} \int_{\mathbb{R}^3} \hat{f}\left(\frac{\xi+\eta}{2}\right) \hat{f}\left(\frac{\xi-\eta}{2}\right) \\
&\qquad \times \left(\frac{2}{||\xi|^2 - |\eta|^2|^2} - \left(\frac{1}{|\xi-\eta|^4} + \frac{1}{|\xi+\eta|^4} \right) \right) d\eta.
\end{aligned}$$

Furthermore, letting $\lambda \to -1$ we obtain

$$\widehat{T_{-1}}(\xi,\eta) = -16\pi^2 C_{-1}\left(\frac{1}{|\xi-\eta|^2} + \frac{1}{|\xi+\eta|^2} + \frac{\log\big||\xi|-|\eta|\big| - \log\big||\xi|+|\eta|\big|}{|\xi||\eta|}\right),$$

and, finally, for $\lambda \to -2$

$$\widehat{T_{-2}}(\xi,\eta) = -\pi^3 C_{-2}\left(\frac{2}{\min(|\xi|,|\eta|)} + \frac{1}{|\xi-\eta|} + \frac{1}{|\xi+\eta|}\right).$$

3.2 The case $\lambda = 0$

The kernel $\widehat{T_0}$ cannot be obtained by simply letting $\lambda \to 0$ in (24). Instead we compute

$$\begin{aligned}
\widehat{T_0}(\xi,\eta) &= C_0 \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}(y,\eta)} \int_{S^2} (\mathrm{e}^{-\mathrm{i}(|y|e,\xi)} + \mathrm{e}^{\mathrm{i}(|y|e,\xi)} - \mathrm{e}^{-\mathrm{i}(y,\xi)} - \mathrm{e}^{\mathrm{i}(y,\xi)}) dy de \\
&= C_0\left(8\pi \int_{\mathbb{R}^3} \frac{\sin(|y||\xi|)}{|y||\xi|} \mathrm{e}^{\mathrm{i}(y,\eta)} dy - 32\pi^4(\delta(\xi-\eta)+\delta(\xi+\eta))\right) \\
&= C_0\left(\frac{8\pi}{|\xi|}\int_0^\infty r\sin(r|\xi|)\int_{S^2} \mathrm{e}^{i(re_y,\eta)} de_y dr - 32\pi^4(\delta(\xi-\eta)+\delta(\xi+\eta))\right) \\
&= C_0\left(\frac{32\pi^2}{|\eta||\xi|}\int_0^\infty \sin(r|\xi|)\sin(r|\eta|) dr - 32\pi^4(\delta(\xi-\eta)+\delta(\xi+\eta))\right) \\
&= C_0\left(\frac{16\pi^3}{|\eta||\xi|}(\delta(|\xi|-|\eta|) - \delta(|\xi|+|\eta|)) - 32\pi^4(\delta(\xi-\eta)+\delta(\xi+\eta))\right).
\end{aligned}$$

Inserting this into the transformed equation, we easily obtain the following special case of Bobylev's classical result (7):

$$\hat{f}_t(\xi) = C_0 \int_{S^2} \left(\hat{f}\left(\frac{\xi+|\xi|e}{2}\right)\hat{f}\left(\frac{\xi-|\xi|e}{2}\right) - \hat{f}(\xi)\hat{f}(0)\right) de. \tag{25}$$

Remark 2 Of course, one could derive the collision invariants from the explicit expressions above. But this is already done quite easily by use of the general form of the kernel in (18) (see Remark 1).

4 Concluding Remarks

Under weak assumptions on f and the collision kernel B, it is possible to find a quite simple representation of the Boltzmann equation in terms of $\hat{f}$.

Maybe theoretical studies of the initial value problem can benefit from the simple structure of the transformed equation, but in any case, there is the following impact for numerical studies of the Boltzmann equation: Since the operator $\mathcal{J}(\hat{f},\hat{f})$ in (21) contains only one three-dimensional integral, even the most straightforward approximation method will lead to a numerical scheme with competitive efficiency.

Moreover, the kernel $\widehat{T_\lambda}$ in (18) is explicitly known for the practically relevant VHS models for all $\lambda \in (-3, 1]$ but the singularities appearing in the kernel are quite strong. However, numerical methods can be constructed either by truncation of the collision kernel

B in velocity space (cf. e.g. [20]), or by using a rapidly decreasing version of the collision kernel, e.g.

$$B_{\lambda,\sigma}(u) = C_\lambda |u|^\lambda \mathrm{e}^{-\frac{\sigma}{8}|u|^2}, \quad \sigma > 0$$

for which it is possible to obtain a smooth version of the kernel $\widehat{T_\lambda}$, given by

$$\begin{aligned}\widehat{T_{\lambda,\sigma}}(\xi,\eta) = C_{\lambda,\sigma}\Bigg(\frac{1}{|\xi||\eta|}\Bigg({}_1F_1\Bigg(\frac{\lambda+1}{2},\frac{1}{2},-\frac{(|\xi|-|\eta|)^2}{2\sigma}\Bigg) \\ - {}_1F_1\Bigg(\frac{\lambda+1}{2},\frac{1}{2},-\frac{(|\xi|+|\eta|)^2}{2\sigma}\Bigg)\Bigg) \\ - \frac{\lambda+1}{\sigma}\Bigg({}_1F_1\Bigg(\frac{\lambda+3}{2},\frac{3}{2},-\frac{|\xi-\eta|^2}{2\sigma}\Bigg) + {}_1F_1\Bigg(\frac{\lambda+3}{2},\frac{3}{2},-\frac{|\xi+\eta|^2}{2\sigma}\Bigg)\Bigg)\Bigg),\end{aligned}$$

where ${}_1F_1$ denotes the Kummer confluent hypergeometric function and the constant $C_{\lambda,\sigma}$ is defined as follows

$$C_{\lambda,\sigma} = 8\pi^2 C_\lambda \Gamma\left(\frac{\lambda+1}{2}\right)\left(\frac{2}{\sigma}\right)^{\frac{\lambda+1}{2}}.$$

The construction and study of numerical schemes based on this approach for small $\sigma \to 0$ shall be presented in a forthcoming paper.

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